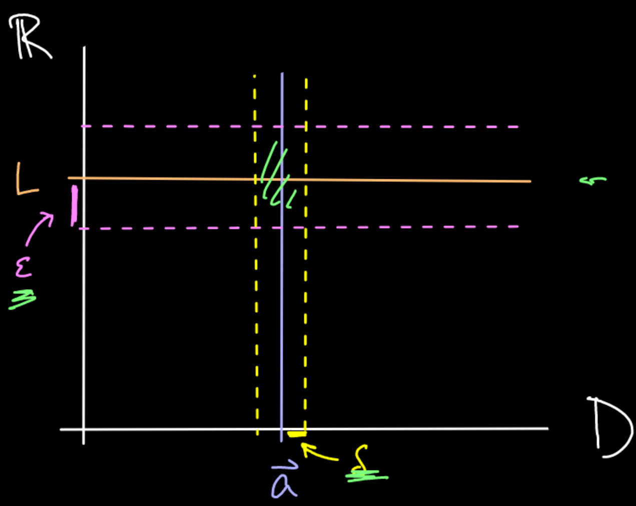
Limits of Multivariate Functions

Goal: Understand limits of functions of several variables.

Def^n: Let $f : D \to \mathbb{R}$ have $D \subseteq \mathbb{R}^n$ and let $\vec{a} \in \mathbb{R}^n$. Real $L \in \mathbb{R}$ is the limit of f as $\vec{x}$ approaches $\vec{a}$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{x} \in \mathbb{R}^n$ we have $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$.



NB: This is the SAME definition as from Calculus I...

The only difference: now we allow $D \subseteq \mathbb{R}^n$ for $n > 1$...

Notation: $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ OR $f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

NB: In practice, we won't use the formal definition of the limit in this course...

Instead, we will will bootstrap our previous work...

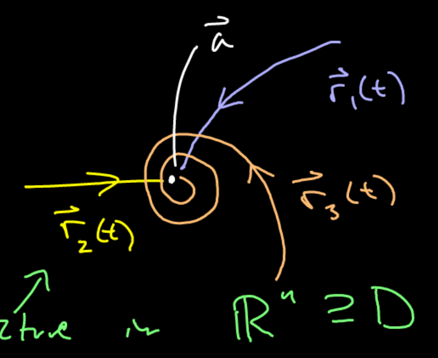
Idea: Rather than work with ε's an δ's, let's use our knowledge of space curves to our advantage!

Prop (Curves Criterion for Limits): Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function, $\vec{a} \in \mathbb{R}^n$, and $L \in \mathbb{R}$. The following are equivalent:

1. We have $f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$ (i.e. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$)
2. For all continuous curves $\vec{r} : [\alpha, \beta) \to \mathbb{R}^n$ with $\lim_{t \to \beta^-} \vec{r}(t) = \vec{a}$ and $\vec{a} \notin \vec{r}[\alpha, \beta)$ we have $f(\vec{r}(t)) \to L$ as $t \to \beta^-$.

Point: We can turn multivariate limits into single variable limits, but we must pay the price of checking limits over all curves...

Picture



Note that this condition doesn't (at first glance) make computing limits any easier... It does however make a difference to showing a limit does not exist! Just find two curves along which the function limits differently!

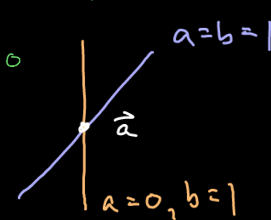
Ex: Show $\lim_{\vec{x} \to \vec{0}} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

straight lines through $\vec{0}$ in $\mathbb{R}^2$.

Sol: Consider $\vec{l}_{a,b}(t) = \langle at, bt \rangle$ for $a, b \in \mathbb{R}$ (not both 0).

$f(at, bt) = \frac{(at)^2 - (bt)^2}{(at)^2 + (bt)^2} = \frac{(a^2 - b^2)t^2}{(a^2 + b^2)t^2} = \frac{a^2 - b^2}{a^2 + b^2}$ when $t \neq 0$

$\vec{l}_{a,b}(t) \to \vec{0}$ as $t \to 0$



$\therefore$ If the limit exists, the curves criterion implies

$L = \lim_{\vec{x} \to \vec{0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{t \to 0} f(at, bt) = \lim_{t \to 0} \frac{a^2 - b^2}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2}$ for all $a, b \in \mathbb{R}$

But if $a = b = 1$, we obtain $L = 0$ while for $a = 0, b = 1$ we obtain $L = -1$

Hence the limit does not exist (otherwise $0 = L = -1$) $\boxed{↯}$

NB: The above solution used the lines with formulas

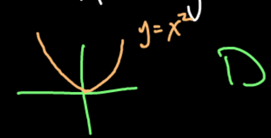
$$\vec{l}_{a,b}(t) = \langle at, bt \rangle \quad \text{for } a,b \in \mathbb{R} \quad (\text{not both } 0)$$

We can cleverly shift this collection of lines around to show even more limits don't exist.

WARNING: Just because the collection of lines above all yield the same limit DOES NOT mean the limit exists.

Ex: Consider the piecewise function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \leftarrow \\ 0 & \text{otherwise} \end{cases}.$$



Along the lines $\vec{l}_{a,b}(t) = \langle at, bt \rangle$ we have (for $t \neq 0$):

$$f(at, bt) = 1 \iff bt = (at)^2 = a^2t^2 \iff b = a^2 t.$$

Thus, if $a \neq 0$ we have $t = \frac{b}{a^2}$. Hence

$$\lim_{t \to 0} f(\vec{l}_{a,b}(t)) = \lim_{t \to 0} 0 = 0 \quad \text{for all } a,b \in \mathbb{R}.$$

Along the curve $\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle$ we have $f(\vec{r}(t)) = 1$, so

$$\lim_{t \to 0} f(\vec{r}(t)) = \lim_{t \to 0} 1 = 1.$$

Hence $\lim_{\vec{x} \to \vec{0}} f(\vec{x})$ does not exist by the Curves Criterion for Multivariate Limits.

Ex: Does $\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+y^2-2x+1}$ exist?

Sol: (**Trick**: Apply curves criterion using lines passing through $(1,0)$)

Consider the curves $\vec{l}_{a,b}(t) = \langle at, bt\rangle + \langle 1, 0\rangle = \langle at+1, bt\rangle$

$$f(at+1, bt) = \frac{(at+1)bt - bt}{(at+1)^2 + (bt)^2 - 2(at+1)+1} = \frac{abt^2}{a^2t^2 + \cancel{2at} + \cancel{1} + b^2t^2 - \cancel{2at} - \cancel{2} + \cancel{1}}$$

$$= \frac{abt^2}{(a^2+b^2)t^2} = \frac{ab}{a^2+b^2} \quad \text{when } t \neq 0$$

for $a=0, b=1$ $f(\vec{l}_{0,1}(t)) = 0$ while for $a=b=1$ $f(\vec{l}_{1,1}(t)) = \frac{1}{2}$

Hence $\lim_{t\to0} f(\vec{l}_{0,1}(t)) = 0 \neq \frac{1}{2} = \lim_{t\to0} f(\vec{l}_{1,1}(t))$, so the limit does not exist by the curves criterion. ▨

Ex: Does $\lim_{\vec{x}\to\vec{0}} \frac{xy+yz}{x^2+y^2+z^2}$ exist?

Sol: Consider the curves $\vec{l}_{a,b,c}(t) = \langle a, b, c\rangle t$.
(i.e. lines passing through the origin as $t \to 0$)

CHECK!

$f(x,y,z) = \frac{xy+yz}{x^2+y^2+z^2}$ has $f(\vec{l}_{a,b,c}(t)) = \frac{b(a+c)}{a^2+b^2+c^2}$ when $t \neq 0$.

(Next we'll find two lines with different limiting values ...)

Thus we have the following:

$\lim_{t\to0} f(\vec{l}_{1,0,1}(t)) = \lim_{t\to0} 0 = 0$ whereas $\lim_{t\to0} f(\vec{l}_{1,1,1}(t)) = \lim_{t\to0} \frac{2}{3} = \frac{2}{3}$

(Now we can conclude that $\lim_{\vec{x}\to\vec{0}} f(x,y,z)$ does not exist ...)

Hence $\lim_{\vec{x}\to\vec{0}} \frac{xy+yz}{x^2+y^2+z^2}$ does not exist, by the Curves Criterion. ▨

Continuity of Multivariate Functions

Goal: Extend the definition of continuity to multivariate functions.

Def^n: A function $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>continuous</u> at $\vec{a} \in \mathbb{R}^n$ when $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$. The function f is <u>continuous</u> on $S \subseteq \mathbb{R}^n$ when f is continuous at all $\vec{a} \in S$.

NB: That's the same as Calculus I... ✱

Here's a quick reference for common continuous functions:

- → Polynomials in any number of variables.
  Ex: $x^2 + yz - 7x^3yz^2$
- ✱ → Compositions of continuous functions.
  Ex: $\exp(x^2 + yz)$ $\alpha : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\alpha \circ f$
- → Sums and products of continuous functions.
  $f, g : \mathbb{R}^n \to \mathbb{R}$ cts $\Rightarrow$ $f \cdot g$ and $f + g$
- → Continuous changes of coordinate Ex: Cartesian $\longmapsto$ polar
  (essentially about composition)
- ✱ → The functions you studied in Calculus I and Calculus II.

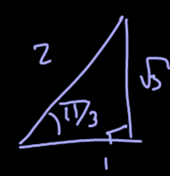
Point: We can use <u>Calculus I</u> and <u>Calculus II</u> again!

Ex: Compute the limit if it exists, or show it doesn't exist.

ⓐ $\lim_{\vec{x} \to (3,2)} (x^2 y^3 - 4y^2)$ ← cts (polynomial)

$= 3^2 \cdot 2^3 - 4 \cdot 2^2 = 72 - 16 = 56$

ⓑ $\lim_{\vec{x} \to (\pi, 0, 1/3)} e^{y^2} \cdot \tan(xz)$ (product; polynomials; composition) ∴ cts on its domain!



$= e^{0^2} \cdot \tan(\pi \cdot \frac{1}{3}) = \tan(\frac{\pi}{3}) = \sqrt{3}$

ⓒ $\lim_{\vec{x} \to \vec{0}} \frac{x^3 + y^3}{x^2 + y^2}$

(TRICK: Change to polar coordinates)

$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

$(x,y) \to \vec{0}$ as $r \to 0^+$

$= \lim_{r \to 0^+} \frac{(r\cos(\theta))^3 + (r\sin(\theta))^3}{(r\cos(\theta))^2 + (r\sin(\theta))^2}$

$= \lim_{r \to 0^+} \frac{r^3\cos^3(\theta) + r^3\sin^3(\theta)}{r^2\cos^2(\theta) + r^2\sin^2(\theta)}$

$= \lim_{r \to 0^+} \frac{r^3(\cos^3(\theta) + \sin^3(\theta))}{r^2(\cos^2(\theta) + \sin^2(\theta))}$

$= \lim_{r \to 0^+} r(\cos^3(\theta) + \sin^3(\theta))$

$= 0$

Squeeze Theorem using $-2r \leq r(\cos^3(\theta) + \sin^3(\theta)) \leq 2r$ yields ...

(d) $$\lim_{\vec{x}\to(0,0)} \frac{xy^4}{x^4+y^4}$$

$$= \lim_{r\to 0^+} \frac{r^5 \cos(\theta)\sin^4(\theta)}{r^4\left(\cos^4(\theta)+\sin^4(\theta)\right)}$$

$$= \lim_{r\to 0^+} r\left(\frac{\cos(\theta)\sin^4(\theta)}{\cos^4(\theta)+\sin^4(\theta)}\right) \leftarrow \text{apply squeeze theorem!}$$

$$= 0$$

$\vec{l}_{a,b}(t) = \langle a, b\rangle t$

$f(at, bt) = \frac{ab^4 t}{a^4+b^4}$ ← check!

limits to 0 along these lines...

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$r\to 0^+ \iff (x,y)\to\vec{0}$

(e) $$\lim_{\vec{x}\to\vec{0}} (x^2+y^2)\ln(x^2+y^2)$$

$$= \lim_{r\to 0^+} \left(r^2\cos^2(\theta)+r^2\sin^2(\theta)\right)\ln\left(r^2\cos^2(\theta)+r^2\sin^2(\theta)\right)$$

$$= \lim_{r\to 0^+} r^2 \ln(r^2)$$

$$= \lim_{r\to 0^+} \frac{2\ln(r)}{r^{-2}} \leftarrow \text{indeterminate form of type } -\frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{r\to 0^+} \frac{2/r}{-2r^{-3}}$$

$$= \lim_{r\to 0^+} -r^3/r$$

$$= \lim_{r\to 0^+} -r^2 = 0$$

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$\vec{x}\to\vec{0} \iff r\to 0^+$

▨

Ex: Compute the set of points on which the function is continuous.

ⓐ $f(x,y) = \sqrt{y - x^2}\,\ln(x+1)$

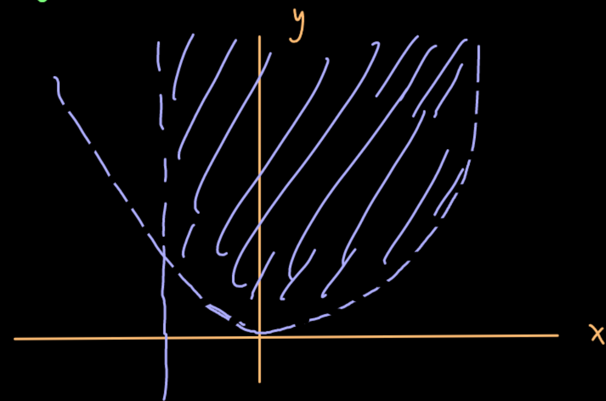
Sol: (annotation: $\sqrt{y-x^2}$ and $\ln(x+1)$ ← cts on their domain)

$\ln(x+1)$ is cts when $x+1 > 0$ (i.e. $x > -1$)

$\sqrt{y-x^2}$ is cts when $y - x^2 > 0$ (i.e. $y > x^2$)

$\therefore$ $f(x,y)$ is cts on

$\{(x,y) : x > -1,\ y > x^2\} \subseteq \mathbb{R}$



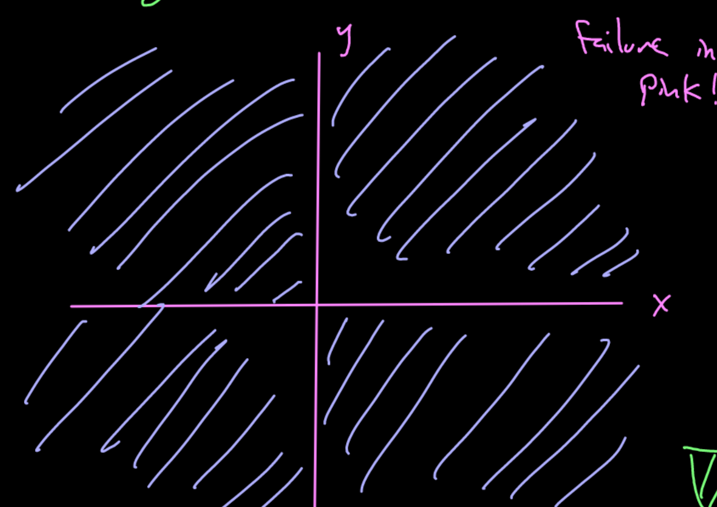
ⓑ $g(x,y) = \dfrac{e^x + e^y}{e^{xy} - 1}$ ← rational function

$g(x,y)$ is cts for all (x,y) w/ $e^{xy} - 1 \neq 0$

i.e. $e^{xy} \neq 1$ i.e. $xy \neq 0$ i.e. $x \neq 0$ AND $y \neq 0$

$\therefore$ the desired set

is $\{(x,y) : x, y \neq 0\} \subseteq \mathbb{R}^2$



◪

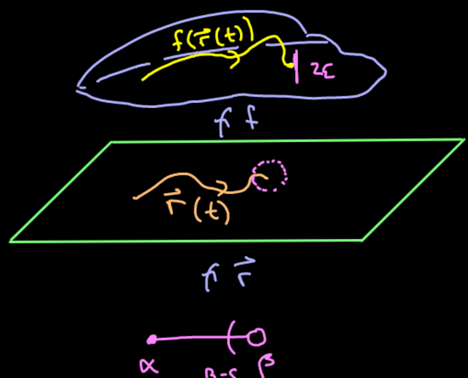
Prop (Curves Criterion for Limits): Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function, $\vec{a} \in \mathbb{R}^n$, and $L \in \mathbb{R}$. The following are equivalent:

① We have $f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

② For all continuous curves $\vec{r} : [\alpha, \beta) \to \mathbb{R}^n$ with $\lim_{t \to \beta^-} \vec{r}(t) = \vec{a}$ and $\vec{a} \notin \vec{r}[\alpha, \beta)$ we have $f(\vec{r}(t)) \to L$ as $t \to \beta^-$.

pf: Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$ be arbitrary.

①⇒②: Suppose $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$. Let $\vec{r} : [\alpha, \beta) \to \mathbb{R}^n$ satisfy $\lim_{t \to \beta^-} \vec{r}(t) = \vec{a}$ and $\vec{a} \notin \vec{r}[\alpha, \beta)$. (WTS (want to show): $\lim_{t \to \beta^-} f(\vec{r}(t)) = L$).

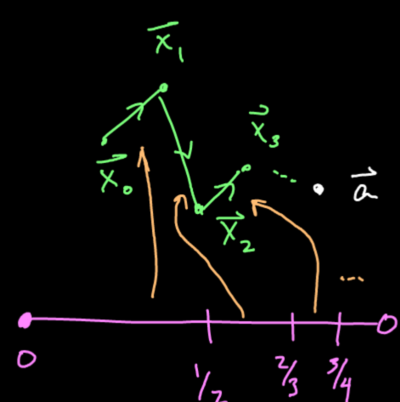


Let $\varepsilon > 0$ be arbitrary. By assumption, there is a $\delta_1 > 0$ s.t. $|\vec{x} - \vec{a}| < \delta_1$ implies $|f(\vec{x}) - L| < \varepsilon$. On the other hand, $\lim_{t \to \beta^-} \vec{r}(t) = \vec{a}$ implies there is a $\delta > 0$ such that $t \in (\beta - \delta, \beta)$ implies $|\vec{r}(t) - \vec{a}| < \delta_1$, and thus we have $t \in (\beta - \delta, \beta)$ implies $|f(\vec{r}(t)) - \vec{a}| < \varepsilon$. Hence $\lim_{t \to \beta^-} f(\vec{r}(t)) = L$.

②⇒①: Suppose $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) \neq L$. We will construct a path $\vec{r}(t)$ which fails the second condition. (NB: We're proving "① fails implies ② fails").

Because $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) \neq L$, there is an $\varepsilon > 0$ s.t. for all $\delta > 0$ there is an $\vec{x} \in D \setminus \{\vec{a}\}$ with $|\vec{x} - \vec{a}| < \delta$ but $|f(\vec{x}) - L| \geq \varepsilon$.

(NB: I'm implicitly assuming there is a ball B surrounding $\vec{a}$ with f defined at every element of B... If not, we can do a similar trick to the below with nondomain points...)



For $\delta_k = \frac{1}{k}$ (where $k > 0$ is an integer), choose an $\vec{x}_k \in D$ such that $0 < |\vec{x}_k - \vec{a}| < \frac{1}{k}$ but $|f(\vec{x}_k) - L| \geq \varepsilon$. To construct $\vec{r}(t)$, let $\vec{r}(1 - \frac{1}{k}) = \vec{x}_k$ for each positive integer k, and extend the curve by interpolating along line segments connecting $\vec{x}_k$ to $\vec{x}_{k+1}$. The result is a continuous curve $\vec{r} : [0, 1) \to \mathbb{R}^n$ with $\lim_{t \to 1^-} \vec{r}(t) = \vec{a}$ but $\lim_{t \to 1} f(\vec{r}(t)) \neq L$. Hence condition ② fails.

Hence we may conclude the original statement is true. ///

Prop: The functions $f : \mathbb{R}^n \to \mathbb{R} : \vec{x} \mapsto |\vec{x}|$ and $g : \mathbb{R}^n \to \mathbb{R} : \vec{x} \mapsto \vec{c} \cdot \vec{x}$ are continuous on $\mathbb{R}^n$ (where $\vec{c} \in \mathbb{R}^n$ is fixed in advance...).

pf: ① $\vec{x} \cdot \vec{x}$ is a polynomial function, so it is continuous on $\mathbb{R}^n$. Moreover $\vec{x} \cdot \vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$. Hence $\sqrt{\vec{x} \cdot \vec{x}} = |\vec{x}|$ is a cts function on $\mathbb{R}^n$.

② For a fixed $\vec{c} \in \mathbb{R}^n$, $\vec{c} \cdot \vec{x}$ is a polynomial. ☑